Multidimensional cut-off technique, odd-dimensional Epstein zeta functions and Casimir energy of massless scalar fields

Ariel Edery

Physics Department, Bishop's University, Lennoxville, Quebec J1M1Z7, Canada

E-mail: aedery@ubishops.ca

Abstract
Quantum fluctuations of massless scalar fields represented by quantum fluctuations of the quasiparticle vacuum in a zero-temperature dilute Bose–Einstein condensate may well provide the first experimental arena for measuring the Casimir force of a field other than the electromagnetic field. This would constitute a real Casimir force measurement—due to quantum fluctuations—in contrast to thermal fluctuation effects. We develop a multidimensional cut-off technique for calculating the Casimir energy of massless scalar fields in d-dimensional rectangular spaces with q large dimensions and $d - q$ dimensions of length L and generalize the technique to arbitrary lengths. We explicitly evaluate the multidimensional remainder and express it in a form that converges exponentially fast. Together with the compact analytical formulae we derive, the numerical results are exact and easy to obtain. Most importantly, we show that the division between analytical and remainder is not arbitrary but has a natural physical interpretation. The analytical part can be viewed as the sum of individual parallel plate energies and the remainder as an interaction energy. In a separate procedure, via results from number theory, we express some odd-dimensional homogeneous Epstein zeta functions as products of one-dimensional sums plus a tiny remainder and calculate from them the Casimir energy via zeta function regularization.

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1. Introduction

The Casimir force remained for a long time one of the most esoteric forces in physics attracting at best some theoretical interest. All of this has changed in the last eight years or so. After nearly 50 years since its prediction in 1948 by Casimir [1], the force has now

been successfully measured by a modern series of experiments starting with Lamoreaux's 1997 landmark experiment [3] with a torsion pendulum which reduced errors dramatically compared to the early 1958 experiment by Spaarnay [2]. The force was subsequently measured more precisely in 1998 using an atomic force microscope [4] and the measurements agreed with theoretical predictions to within 1% after finite conductivity, roughness and temperature corrections were taken into account. Thus the modern era of precise Casimir measurements was born and a non-exhaustive list of other experimental studies since then can be found in [5–13]. Interest in the Casimir force has also been fuelled by theories with large extra dimensions which predict among other things a deviation from Newtonian gravitation at the sub-millimetre scale [14]. To date no deviation has been found. Recently, a Casimir force experiment [15] has placed new constraints on the parameters of such proposed theories. An up-to-date list of gravitational experiments can also be found in [15]. As with many fundamental physics discoveries, at first the Casimir force seemed to have no apparent engineering application (since it is significant only on micron or submicron scales). However, our ever increasing ability to build structures on smaller scales has made the Casimir force something various industries need to take into account. For example, in 2001, scientists at Lucent Technologies showed that the Casimir force could be used to control the mechanical motion of a microelectromechanical system (MEMS) device [17] (see also the recent paper [16] and references therein). MEMS are micron-sized devices in which tiny sensors and actuators are carved into a silicon substrate and are currently in use as car air-bag sensors. For more details on the Casimir effect the reader is referred to the following books [28, 29] and reviews [30–33].

All the measurements of the Casimir force to date have been limited to the case of the electromagnetic field. However, experiments may soon (or may already have done so indirectly) measure the Casimir force for a massless scalar field. Quantum fluctuations of the quasiparticle vacuum in a zero-temperature dilute Bose–Einstein condensate (BEC) should give rise to a measurable Casimir force as explained in recent papers [18, 19]. The authors in [18, 19] state that indirect effects from these quantum fluctuations may have already been observed [20–22, 27]. Note that this is a real Casimir effect due to quantum fluctuations in contrast to thermal fluctuations (often called pseudo-Casimir). The fact that the field propagates at the speed of sound in the BEC medium in contrast to the speed of light in Minkowski spacetime does not change anything fundamental in relation to the Casimir energy. If the speed of propagation is constant in a given medium, the Casimir energy in units of this speed will be the same value regardless of whether the medium is spacetime or a BEC. Moreover, a generally covariant action analogous to what we see in general relativity exists for scalar fields propagating in a particular fluid. The Lagrangian is similar to that of a massless Klein–Gordon field with the Minkowski metric $\eta_{\mu\nu}$ of spacetime replaced by an effective or acoustic metric $g_{\mu\nu}$ [34]. Quoting directly from [35], 'at low momenta linearized excitations of the phase of the condensate wavefunction obey a (3+1)-dimensional d'Alembertian equation coupling to a (3+1)-dimensional Lorentzian-signature "effective metric" that is generic, and depends algebraically on the background field'. In [19] the authors make the important observation that though the dispersion relation for quantum fluctuations in a BEC is nonlinear, the Casimir energy picks out mostly the long wavelength linear behaviour. This is why the Casimir force F_{BEC} calculated by the same authors [18, 19] for infinitely thin and infinitely repulsive plates immersed in a zero-temperature three-dimensional dilute condensate turns out to leading order to be the same as that of a massless scalar field moving with the speed of sound v.

In this paper, we are interested in the Casimir effect of massless scalar fields travelling with speed v in rectangular cavities of d spatial dimensions where q dimensions are large and

$d - q$ dimensions are of equal length L. The case of arbitrary lengths is also considered in appendix B. We develop a multidimensional cut-off technique to solve this problem. Why use a cut-off technique? Clearly, it is less efficient than the zeta function technique that yields quickly, via analytic continuation, finite results for rectangular cavities in terms of Epstein zeta functions. There are a few reasons for the importance of the exponential cut-off technique. First, it remains the most physically intuitive method. For this reason, recent texts in string theory or quantum field theory (QFT) as well as courses in QFT introduce the standard Casimir energy calculation of a string or parallel plates using an exponential cut-off. For example, in the text *String Theory, vol I* [23], the Casimir energy for the Bosonic string is handled with an exponential cut-off. The result $\sum_{n=1}^{\infty} n \to \frac{-1}{12}$ is obtained by replacing n by $n\,e^{-\lambda n}$ and extracting the finite result $\frac{-1}{12}$ from the series $\frac{1}{\lambda^2} - \frac{1}{12} + O(\lambda^2)$. This cut-off method was used instead of the zeta function technique which yields quickly $\zeta(-1) = \frac{-1}{12}$. In his recent book, *quantum field theory in a nutshell* [24], Zee brings in some humour in explaining a physicist's perspective on the same sum. I quote from p 66, 'Aagh! What do we do with $\sum_{n=1}^{\infty} n$? None of the ancient Greeks from Zeno on could tell us. What they should tell us is that we are doing Physics . . . Physical plates cannot keep arbitrarily high frequencies from leaking out'. He then introduces the exponential cut-off to damp the ultraviolet frequencies. In the classic QFT text by Itzykson and Zuber [25] the electromagnetic parallel plate problem in three dimensions is solved via a cut-off function and the Euler–Maclaurin formula and the same technique can be seen applied in recent graduate courses (e.g. see 'Relativistic Quantum Field Theory I, Spring 2003' [26]). Physicists are therefore likely to be familiar with the cut-off technique. Secondly, a multidimensional cut-off calculation with an exact determination of the multidimensional remainder term does not seem to have been systematically carried out for rectangular cavities in arbitrary d dimensions. Papers on Casimir energies in arbitrary d dimensions in rectangular cavities have made use of dimensional and zeta function regularization [42–44]. Explicit formulae using the exponential cut-off technique in rectangular cavities include parallel plates in higher dimensions [36], rectangular cavities in two and three dimensions [37–40] and explicit formulae via Poisson's formula up to $d = 2$ appear in [41]. In [36, 37, 41] the connection between cut-off and zeta function technique is also elaborated and explained. A detailed numerical analysis for the electromagnetic case in three-dimensional rectangular cavities can be found in [45]. Last but not least, by applying the cut-off technique to rectangular cavities we are led in a natural fashion to excellent finite analytical formulae plus a remainder. We show that the division between analytical and remainder is not some ad hoc division. The analytical part has a clear physical interpretation as sums of parallel plates out of which the rectangular cavity is constructed. Moreover, the numerical results are excellent because the analytical part is trivial to evaluate and the multidimensional remainder is derived in a form that converges quickly (exponentially fast). As already mentioned, the zeta function technique applied to rectangular spaces has the great advantage of leading quickly to finite results expressed in terms of Epstein zeta functions. However, one then needs to go a few steps further if one wants to express these in a convenient analytical form and this is usually a separate procedure. In contrast, analytical results are often a natural spin-off of the cut-off technique.

One section of this paper is devoted to developing a technique that derives highly accurate analytical formulae for a few odd-dimensional homogeneous Epstein zeta functions. It turns out that in even dimensions less than or equal to 8 one can obtain compact analytical expressions for the homogeneous Epstein zeta function purely in terms of products of one-dimensional sums. There is no remainder for these cases. This can be accomplished via number theoretic formulae for the representation of integers as a sum of squares in even dimensions. For even dimensions above 8, the number theoretic formulae get more

complicated and in odd dimensions above 7 they are not presently known. For 3, 5 and 7 dimensions, the number theoretic formulae have only recently been found [50] but they are much more complicated than in even dimensions. We therefore develop a procedure that uses the exact even-dimensional results from number theory and then apply the Euler–Maclaurin formula to obtain the odd dimensions. This yields the homogeneous Epstein zeta function in 3, 5 and 7 dimensions as a finite number of products of one-dimensional sums plus a small remainder term. This remainder is even smaller than the remainder obtained via our multidimensional cut-off technique. For the most important case of 3 dimensions, we obtain both a highly compact and extremely accurate analytical expression that contains only four terms and where the remainder is a negligible 0.04% of the Casimir energy. Our specific procedure leads to low remainders but is limited to a few homogeneous Epstein zeta functions, albeit one that includes the three-dimensional case. A different more general procedure applicable to any multidimensional inhomogeneous Epstein-type zeta function can be found in [46].

2. Multidimensional cut-off technique including remainder

In this section, we develop a multidimensional cut-off technique to obtain formulae for the Casimir energy of a massless scalar field $\phi(x)$ moving with a wave velocity v in a d-dimensional rectangular cavity with $d - q$ sides of equal length L and q sides of much larger length $L_m \gg L$ where m runs from 1 to q. One can generalize our method to arbitrary lengths and this is done in appendix B. Here and throughout the paper we consider the more special case as it makes the method, the formulae and the physical interpretation more transparent. This section and appendix A (where the remainder is evaluated) go together.

We consider periodic, Neumann and Dirichlet boundary conditions. The fields are assumed to propagate in a homogeneous medium with a constant speed v and with a wavelength long enough that the dispersion relation is linear, i.e. $\omega = vk$ where k is the wave number. In other words, we assume the scalar field $\phi(x)$ to obey the standard linear wave equation:

$$\frac{\partial^2 \phi(x)}{\partial^2 t} - v^2 \nabla^2 \phi(x) = 0. \tag{1}$$

The boundary conditions are either periodic, $\phi(x^i = 0) = \phi(x^i = L)$, Neumann, $\partial^i \phi(x) = 0$ at $x^i = 0$ and $x^i = L$ or Dirichlet $\phi(x^i = 0) = \phi(x^i = L) = 0$. Here i runs from 1 to $d - q$ inclusively. After the standard Fourier decomposition one obtains the following quantized frequencies ω for periodic (p), Neumann (N) and Dirichlet (D) conditions:

$$
\begin{aligned}
\omega_{\mathrm{p}} &= 2\pi v \left(\frac{n_1^2}{L^2} + \cdots + \frac{n_{d-q}^2}{L^2} + \frac{n_{d-q+1}^2}{L_1^2} + \cdots + \frac{n_d^2}{L_q^2} \right)^{1/2} \\
\omega_{\mathrm{N,D}} &= \pi v \left(\frac{n_1^2}{L^2} + \cdots + \frac{n_{d-q}^2}{L^2} + \frac{n_{d-q+1}^2}{L_1^2} + \cdots + \frac{n_d^2}{L_q^2} \right)^{1/2},
\end{aligned}
\tag{2}
$$

where n_i's run from $-\infty$ to ∞ for periodic boundary conditions, 0 to ∞ for Neumann and 1 to ∞ for Dirichlet. From quantum field theory, we know that after quantization the vacuum energy is given by the sum over all modes of $\frac{1}{2}\omega$ (we work in units where $\hbar = 1$). The vacuum

energies E^{vac} for the three boundary conditions labelled (p, N, D) are therefore,

$$
E_{\mathrm{p}}^{\mathrm{vac}} = \frac{\pi v}{L} \sum_{\substack{n_i=-\infty \\ i=1,\dots,d}}^{\infty} \left(n_1^2 + \cdots + n_{d-q}^2 + \frac{n_{d-q+1}^2 L^2}{L_1^2} + \cdots + \frac{n_d^2 L^2}{L_q^2} \right)^{1/2},
$$

$$
E_{\mathrm{N,D}}^{\mathrm{vac}} = \frac{\pi v}{2L} \sum_{\substack{n_i=0,1 \\ i=1,\dots,d}}^{\infty} \left(n_1^2 + \cdots + n_{d-q}^2 + \frac{n_{d-q+1}^2 L^2}{L_1^2} + \cdots + \frac{n_d^2 L^2}{L_q^2} \right)^{1/2}.
$$

$$(3)$$

The above sums are ultraviolet divergent and require regularization. There are many different regularization schemes such as exponential cut-off, zeta function and dimensional regularization. In this paper, the goal is to develop a multidimensional cut-off technique via the Euler–Maclaurin formula. Using this technique, we obtain formulae for the Casimir energy as a finite sum over analytical terms plus a remainder. We fully evaluate the remainder term and express it as sums over Bessel functions. We later show that the analytical part has an intuitive physical picture; it is the energy needed to construct the rectangular cavity out of adding successive parallel plates. We begin by calculating the regularized vacuum energy for periodic boundary conditions. After regularization, we then extract the finite Casimir energy E_{p} which is the difference between the regularized energy with boundaries (discrete modes) and the regularized energy without boundaries (continuous modes). We later compare E_{p} to the Epstein zeta function obtained via zeta function regularization. We can express the Neumann and Dirichlet energies, E_{N} and E_{D}, in terms of sums over E_{p}, so only the periodic case needs to be evaluated fully. The regularized vacuum energy E^{reg} for periodic boundary conditions using an exponential cut-off is

$$
E_{\mathrm{p}}^{\mathrm{reg}}(q, \lambda) = \frac{\pi v}{L} \sum_{\substack{n_i=-\infty \\ i=1,\dots,d}}^{\infty} \left(n_1^2 + \cdots + n_{d-q}^2 + \frac{n_{d-q+1}^2 L^2}{L_1^2} + \cdots + \frac{n_d^2 L^2}{L_q^2} \right)^{1/2},
$$

$$
\exp\left(-\lambda \sqrt{ n_1^2 + \cdots + n_{d-q}^2 + \frac{n_{d-q+1}^2 L^2}{L_1^2} + \cdots + \frac{n_d^2 L^2}{L_q^2} } \right)
$$

$$
= -\frac{\pi v}{L^{q+1}} \prod_{i=1}^{q} L_i \partial_\lambda \sum_{\substack{n_i=-\infty \\ i=1,\dots,d-q}}^{\infty} \int_{-\infty}^{\infty}
$$

$$
\times \exp\left(-\lambda \sqrt{ n_1^2 + \cdots + n_{d-q}^2 + x_1^2 + \cdots + x_q^2 } \right) \mathrm{d}x_1 \cdots \mathrm{d}x_q, \tag{4}
$$

where we replaced the sums over the q large dimensions by integration. The parameter λ is a free parameter which we later set to 0. The goal is to evaluate the expression in (4) that includes $d - q$ sums and q integrals. Our procedure will be to express (4) as an expansion over a function Λ and then use the Euler–Maclaurin formula to evaluate this function. Define the following short-hand form for a $(j - q)$-dimensional sum over q integrals:

$$
\sideset{}{^{j-q}}\sum \int^q \equiv \sum_{\substack{n_i=-\infty \\ i=1,\dots,j-q}}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\lambda \sqrt{ n_1^2 + \cdots + n_{j-q}^2 + x_1^2 + \cdots + x_q^2 } \right) \mathrm{d}x_1 \cdots \mathrm{d}x_q, \tag{5}
$$

where j runs from q to $d - 1$ (the case $j = q$ corresponds to no sums, only q integrals). The reader may wonder why we chose a definition with $j - q$ sums instead of just simply j. The reason is that the total number of sums plus integrals is then j and this simplifies things later on.

We define a function Λ by adding one more sum to the above definition:

$$
\Lambda_j(q,\lambda) \equiv \sideset{}{'}\sum \sum^{j-q} \int^q
$$
$$
= \sideset{}{'}\sum_{n=-\infty}^{\infty} \sum_{\substack{n_i=-\infty \\ i=1,\dots,j-q}}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\lambda\sqrt{n^2 + n_1^2 + \cdots + n_{j-q}^2 + x_1^2 + \cdots + x_q^2}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_q,
$$

(6)

where the last sum over n excludes zero. With these definitions, we make the following useful expansion of (4):

$$
\sum_{\substack{n_i=-\infty \\ i=1,\dots,d-q}}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\lambda\sqrt{n_1^2 + \cdots + n_{d-q}^2 + x_1^2 + \cdots + x_q^2}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_q
$$
$$
= \int^q + \sideset{}{'}\sum \int^q + \sideset{}{'}\sum \sum \int^q + \sideset{}{'}\sum \sum^2 \int^q + \cdots + \sideset{}{'}\sum \sum^{d-q-1} \int^q
$$
$$
= \int^q + \sum_{j=q}^{d-1} \sideset{}{'}\sum \sum^{j-q} \int^q = \int^q + \sum_{j=q}^{d-1} \Lambda_j(q,\lambda).
$$

(7)

Substituting (7) into (4) yields the regularized Casimir energy

$$
E_{\mathrm{p}}^{\mathrm{reg}}(q,\lambda) = -\frac{\pi v}{L^{q+1}} \prod_{i=1}^{q} L_i \left(\partial_\lambda \int^q + \sum_{j=q}^{d-1} \partial_\lambda \Lambda_j(q,\lambda) \right).
$$

(8)

In the above expression, we need to separate the divergent part due to the continuum from the finite part related to the Casimir energy as $\lambda \to 0$. The term $\partial_\lambda \int^q$ contains no sums, only multiple integrals. It is immediately clear that this term contributes purely a continuum divergent part as $\lambda \to 0$ and hence makes no contribution to the finite Casimir energy. We now need to find an expression for $\Lambda_j(q,\lambda)$ given by (6) and extract the finite part related to it. To this end, we apply the Euler–Maclaurin formula that converts sums to integrals. The Euler–Maclaurin formula is given by [51],

$$
\sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(x)\,\mathrm{d}x - \frac{1}{2}f(0) - \sum_{p=1}^{s} \frac{1}{(2p)!} B_{2p} f^{(2p-1)}(0) + R_s,
$$

(9)

where $f^{(2p-1)}(0)$ are odd derivatives evaluated at zero and s is a positive integer. The form above for the Euler–Maclaurin formula assumes that the function $f(n)$ and its derivatives are zero at infinity. R_s is the remainder term given by [51]

$$
R_s = -\frac{1}{(2s)!} \int_0^1 B_{2s}(x) \sum_{v=0}^{\infty} f^{2s}(x+v)\,\mathrm{d}x,
$$

(10)

where $B_{2s}(x)$ are Bernoulli functions and $f^{2s}(x+v)$ are even derivatives of f with respect to x.

In applying the Euler–Maclaurin formula to determine $\Lambda_j(q,\lambda)$, the function f in question is the exponential function appearing in (6). Regardless of the value of p, this exponential function has the property that $f^{2p-1}(0)$ is zero for all sums in (6) except the last one over n. A proof of this is given in the appendix of [53]. If $f^{2p-1}(0)$ is zero for all p it follows that the sum from $p=1$ to s in (9) is zero independent of s. This implies that R_s given by (10) has the same value for any given s in the case of our exponential function. This is proven explicitly in

the appendix of [54]. For calculations we can simply choose s equal to 1. Since $f^{2p-1}(0)$ is zero for all sums except the last one, the Euler–Maclaurin formula for those sums reduces to

$$\sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(x)\,dx - \frac{1}{2}f(0) - \frac{1}{2}\int_0^1 B_2(x)\sum_{\nu=0}^{\infty}\frac{d^2}{dx^2}f(x+\nu)\,dx, \quad (11)$$

where $B_2(x) = x^2 - x + 1/6$. The function f in (6) has the property $f(n_i) = f(-n_i)$. The sum over a given n_i can therefore be written as

$$\sum_{n_i=-\infty}^{\infty} f(n_i) = 2\sum_{n_i=1}^{\infty} f(n_i) + f(0)$$

$$= 2\left(\int_0^{\infty} f(x)\,dx - \frac{1}{2}f(0) - \frac{1}{2}\int_0^1 B_2(x)\sum_{\nu=0}^{\infty}\frac{d^2}{dx^2}f(x+\nu)\,dx\right) + f(0)$$

$$= \int_{-\infty}^{\infty} f(x)\,dx - R, \quad (12)$$

where R is a remainder given by

$$R = \int_0^1 B_2(x)\sum_{\nu=0}^{\infty}\frac{d^2}{dx^2}f(x+\nu)\,dx. \quad (13)$$

From (12) we see that each sum in (6), except the last one, can be replaced by an integral minus R. We therefore have the operator prescription $\sum \to \int - R$. Applying the operator $j - q$ times and then inserting the result in (6) yield

$$\sum^{j-q} = \left(\int - R\right)^{j-q} = \int^{j-q} + \sum_{m=1}^{j-q}(-1)^m\binom{j-q}{m}\int^{j-q-m} R^m \quad (14)$$

and

$$\Lambda_j(q,\lambda) \equiv \sum{}'\sum^{j-q}\int^q = \sum{}'\int^j + \sum{}'\sum_{m=1}^{j-q}(-1)^m\binom{j-q}{m}\int^{j-m} R^m$$

$$= 2^{j+1}\sum_{n=1}^{\infty}\int_0^{\infty}\exp\left(-\lambda\sqrt{n^2+x_1^2+\cdots+x_j^2}\right)dx_1\cdots dx_j + R_j(q,\lambda), \quad (15)$$

where $R_j(q,\lambda)$ is a remainder given by

$$R_j(q,\lambda) \equiv \sum_{m=1}^{j-q}\sum_{n=1}^{\infty}(-1)^m 2\binom{j-q}{m}\int^{j-m} R^m. \quad (16)$$

Substituting R given by (13) into (16) yields

$$R_j(q,\lambda) = \sum_{m=1}^{j-q}\sum_{n=1}^{\infty}(-1)^m\binom{j-q}{m}2^{j-m+1}\int_0^{\infty}\int_0^1\prod_{i=1}^m\sum_{\nu_i=0}^{\infty}B_2(x_i)\frac{\partial^2}{\partial x_i}$$

$$(17)$$

$$\exp\left(-\lambda\sqrt{n^2+(x_1+\nu_1)^2+\cdots+(x_m+\nu_m)^2+y_1^2+\cdots+y_{j-m}^2}\right)dx_1\cdots dx_m\,dy_1\cdots dy_{j-m},$$

where the integrations from 0 to 1 and 0 to ∞ are over x's and y's respectively. The function Λ given by (15) contains two terms. The first term leads to the analytical part and the second term $R_j(q,\lambda)$ yields the remainder. In the limit $\lambda = 0$, $R_j(q,\lambda)$ is zero but not its derivative with respect to λ. It is the derivative with respect to λ that enters into the Casimir energy (8). There is therefore a nonzero contribution to the Casimir energy coming from the remainder

term and we fully evaluate it later on. For now, let us evaluate the analytical term in (15). It can be reduced to an infinite sum over the modified Bessel function $K_0(\lambda n)$ which has a useful series expansion. We first note that the integral in (15) can be expressed in terms of the modified Bessel function $K_{\frac{j-1}{2}}(\lambda n)$ [52]:

$$\int_0^\infty \exp\left(-\lambda\sqrt{n^2 + x_1^2 + \cdots + x_j^2}\right) dx_1 \cdots dx_j = -2^{\frac{1-j}{2}} \pi^{\frac{j-1}{2}} \frac{d}{d\lambda}\left(K_{\frac{j-1}{2}}(\lambda n)\left(\frac{n}{\lambda}\right)^{\frac{j-1}{2}}\right). \tag{18}$$

The modified Bessel function $K_{\frac{j-1}{2}}(\lambda n)$ can be expressed as multiple derivatives of $K_0(\lambda n)$ [52]:

$$K_{\frac{j-1}{2}}(\lambda n)\left(\frac{n}{\lambda}\right)^{\frac{j-1}{2}} = (-1)^{\frac{1-j}{2}}\left(\frac{d}{\lambda\,d\lambda}\right)^{\frac{j-1}{2}} K_0(\lambda n). \tag{19}$$

Substituting (19) and (18) into (15) yields $\Lambda_j(q, \lambda)$ as an infinite sum over the modified Bessel function $K_0(\lambda n)$:

$$\Lambda_j(q, \lambda) = 2^{\frac{j+3}{2}} \pi^{\frac{j-1}{2}} (-1)^{\frac{3-j}{2}} \frac{d}{d\lambda}\left(\frac{d}{\lambda\,d\lambda}\right)^{\frac{j-1}{2}} \sum_{n=1}^\infty K_0(\lambda n) + R_j(q, \lambda). \tag{20}$$

The infinite sum over the modified Bessel function $K_0(\lambda n)$ has the following series expansion [52]:

$$\sum_{n=1}^\infty K_0(\lambda n) = \frac{1}{2}\{C + \ln(\lambda/4\pi)\} + \frac{\pi}{2\lambda} + \pi \sum_{m=1}^\infty \left(\frac{1}{\sqrt{\lambda^2 + 4m^2\pi^2}} - \frac{1}{2m\pi}\right). \tag{21}$$

By substituting (21) into (20) we obtain $\Lambda_j(q, \lambda)$ as an analytic expression plus the remainder $R_j(q, \lambda)$:

$$\begin{aligned} \Lambda_j(q, \lambda) &= -\frac{1}{\lambda^j} 2^j \pi^{\frac{j-1}{2}} \Gamma\left(\frac{j+1}{2}\right) + \frac{1}{\lambda^{j+1}} 2^{j+1} \pi^{\frac{j}{2}} \Gamma\left(\frac{j+2}{2}\right) \\ &\quad + \lambda 2^{j+2} \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{j}{2}} \chi_j(\lambda) + R_j(q, \lambda) \end{aligned} \tag{22}$$

where

$$\chi_j(\lambda) \equiv \sum_{m=1}^\infty \frac{1}{(\lambda^2 + 4m^2\pi^2)^{\frac{j+2}{2}}}. \tag{23}$$

To obtain the regularized vacuum energy $E_{\mathrm{p}}^{\mathrm{reg}}(q, \lambda)$ given by (8) we need to evaluate the derivative of Λ:

$$\begin{aligned} \partial_\lambda \Lambda_j(q, \lambda) &= \frac{j}{\lambda^{j+1}} 2^j \pi^{\frac{j-1}{2}} \Gamma\left(\frac{j+1}{2}\right) - \frac{j+1}{\lambda^{j+2}} 2^{j+1} \pi^{\frac{j}{2}} \Gamma\left(\frac{j+2}{2}\right) \\ &\quad + 2^{j+2} \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{j}{2}} \chi_j(\lambda) + \lambda 2^{j+2} \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{j}{2}} \partial_\lambda \chi_j(\lambda) + \partial_\lambda R_j(q, \lambda). \end{aligned} \tag{24}$$

We now take the limit as $\lambda \to 0$ in (24). Note that the first two terms in (24) are divergent in this limit and represent the infinite continuum energy of surface and volume terms respectively. The Casimir energy is the difference between the discrete and continuum case and therefore these two terms need to be subtracted out. We therefore define

$$\partial_\lambda \Lambda_j^{\mathrm{finite}}(q, \lambda) = 2^{j+2} \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{j}{2}} \chi_j(\lambda) + \lambda 2^{j+2} \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{j}{2}} \partial_\lambda \chi_j(\lambda) + \partial_\lambda R_j(q, \lambda). \tag{25}$$

The above terms in the limit $\lambda = 0$ are

$$\lim_{\lambda \to 0} \chi_j(\lambda) = (2\pi)^{-j-2} \zeta(j+2), \qquad \lim_{\lambda \to 0} \partial_\lambda \chi_j(\lambda) = 0 \tag{26}$$

and we define $R_j(q)$ as

$$R_j(q) \equiv \lim_{\lambda \to 0} \partial_\lambda R_j(q, \lambda). \tag{27}$$

Substituting (26) and (27) into (25) we obtain the compact form

$$\lim_{\lambda \to 0} \partial_\lambda \Lambda_j^{\text{finite}}(q, \lambda) = \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j(q). \tag{28}$$

$R_j(q)$ is the multidimensional remainder which contributes to the Casimir energy. This is evaluated in appendix A and the result is

$$R_j(q) = \frac{1}{\pi} \sum_{m=1}^{j-q} 2^{m+1} \binom{j-q}{m} \sum_{n=1}^{\infty} \sum_{\ell_1,\dots,m=1}^{\infty} \frac{n^{\frac{j+1}{2}} K_{\frac{j+1}{2}}\left(2\pi n \sqrt{\ell_1^2 + \cdots + \ell_m^2}\right)}{\left(\ell_1^2 + \cdots + \ell_m^2\right)^{\frac{j+1}{4}}}. \tag{29}$$

Note that $R_j(q)$ is zero for $j = q$. Expression (29) for the remainder is highly convenient. First, it converges rapidly. The Bessel functions decrease rapidly and therefore only the very first few numbers in each sum are needed to reach high accuracy. Secondly, clever algorithms for Bessel functions are well incorporated in many software packages making numerical computation of the remainder easy and accurate. The finite part of (8) in the limit $\lambda = 0$ yields the Casimir energy for the periodic case:

$$E_{\text{p}}(q, d) = -\frac{\pi v}{L^{q+1}} \prod_{i=1}^{q} L_i \sum_{j=q}^{d-1} \lim_{\lambda \to 0} \partial_\lambda \Lambda_j^{\text{finite}}(q, \lambda)$$

$$= -\frac{\pi v}{L^{q+1}} \prod_{i=1}^{q} L_i \sum_{j=q}^{d-1} \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j(q) \tag{30}$$

with $R_j(q)$ given by (29). Equation (30) is the Casimir energy of a massless scalar field moving with velocity v in a d-dimensional rectangular box with periodic boundary conditions where $d - q$ sides have length L and q sides have much larger lengths. Note the convenient break-up into two terms: a finite analytical formula over the well-known Riemann zeta and gamma functions plus a remainder. Since $R_j(q)$ is zero for $j = q$, the sum for the remainder starts at $j = q + 1$ and is therefore nonzero only if $d \geqslant q + 2$, i.e. nonzero only if there are at least two small dimensions on top of the q large dimensions.

We can now readily express the Casimir energies for the Neumann and Dirichlet cases as sums over the periodic ones. In (3), the sums for the periodic case start at $-\infty$, while for Neumann and Dirichlet cases they start at 0 and 1, respectively. We can express the sums from 0 or 1 to ∞ in terms of sums from $-\infty$ to ∞. The functions being summed have the property $f(n) = f(-n)$. We therefore have the relation $\sum_0^\infty f(n) = \frac{1}{2} \sum_{-\infty}^\infty f(n) + \frac{1}{2} f(0)$ which can be expressed as an operator $\sum_0^\infty \to \frac{1}{2}\left(\sum_{-\infty}^\infty +1\right)$. Applying the operator $d - q$ times yields

$$E_{\text{N}}(q, d) \equiv \frac{\pi v}{2L^{q+1}} \prod_{i=1}^{q} L_i \int_0^\infty \left(\sum_0^\infty\right)^{d-q} \to \frac{\pi v}{2L^{q+1}} \prod_{i=1}^{q} L_i \frac{1}{2^d} \int_{-\infty}^\infty \left(1 + \sum_{-\infty}^\infty\right)^{d-q}$$

$$= 2^{-d-1} \frac{\pi v}{L^{q+1}} \prod_{i=1}^{q} L_i \sum_{m=1}^{d-q} \binom{d-q}{m} \int^q \left(\sum_{-\infty}^\infty\right)^m$$

$$= 2^{-d-1} \sum_{m=1}^{d-q} \binom{d-q}{m} E_{\text{p}}(q, q + m). \tag{31}$$

Substituting (30) into (31) yields the Neumann Casimir energy:

$$E_N(q, d) = -2^{-d-1} \frac{\pi v}{L^{q+1}} \prod_{i=1}^{q} L_i$$

$$\sum_{j=q}^{d-1} \sum_{m=j-q+1}^{d-q} \binom{d-q}{m} \left(\Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j(q) \right). \tag{32}$$

For the Dirichlet case, $\sum_1^{\infty} f(n) = \frac{1}{2} \sum_{-\infty}^{\infty} f(n) - \frac{1}{2} f(0)$ and we obtain

$$E_D(q, d) = 2^{-d-1} \sum_{m=1}^{d-q} (-1)^{d-q+m} \binom{d-q}{m} E_p(q, q+m). \tag{33}$$

Substituting (30) into (33) yields the Dirichlet Casimir energy:

$$E_D(q, d) = 2^{-d-1} \frac{\pi v}{L^{q+1}} \prod_{i=1}^{q} L_i$$

$$\sum_{j=q}^{d-1} (-1)^{d+j} \binom{d-q-1}{j-q} \left(\Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j(q) \right). \tag{34}$$

A special case is that of Dirichlet conditions for parallel plates where all sides except one are large, i.e. $q = d - 1$. $R_j(q)$ is then zero and only $j = d - 1$ is summed:

$$E_{||}(d) = -2^{-d-1} \frac{\pi v}{L^d} \prod_{i=1}^{d-1} L_i \Gamma\left(\frac{d+1}{2}\right) \pi^{\frac{-d-3}{2}} \zeta(d+1). \tag{35}$$

The Casimir pressure for the parallel plates is then

$$P_{||}(d) \equiv -\frac{\partial E_{||}}{\partial V} = -\frac{\hbar v d}{(2L)^{d+1}} \Gamma\left(\frac{d+1}{2}\right) \pi^{\frac{-d-1}{2}} \zeta(d+1), \tag{36}$$

where V is the volume $L \prod_{i=1}^{d-1} L_i$ of the parallel plates and we have re-inserted $\hbar$. The result (36) is in agreement with the higher-dimensional parallel plate cut-off calculation of [36] if we set v and L to unity. For three dimensions we set $d = 3$ and obtain

$$P_{||}(3) = -\frac{\pi^2}{480} \frac{\hbar v}{L^4}, \tag{37}$$

where we used the fact that $\zeta(4) = \pi^4/90$. This result is in agreement with the Casimir calculation for quantum fluctuations in a dilute Bose–Einstein condensate at zero temperature that was recently carried out by [18, 19]. As previously mentioned, though the BEC has a nonlinear dispersion relation the Casimir energy only picks out the low-frequency part since the higher frequencies act as a continuum. The low-frequency part is linear and the dispersion relation is equivalent to that of a massless Klein–Gordon field with speed of light replaced by speed of sound. The pressure in (37) is negative implying attraction and decreases to the fourth power of the distance as in the electromagnetic case. In fact, the classic electromagnetic result $-\frac{\pi^2}{240} \frac{\hbar}{L^4}$ for parallel plates can be obtained by multiplying (37) by 2 for two polarizations and setting v equal to 1 for the speed of light.

Equations (30), (32) and (34) for the Casimir energies contain products of the large dimensions L_i which can be arbitrarily large. It is of more physical interest to obtain the energy densities ε which depend on L only. Dividing the Casimir energies by the volume

$V = L^{d-q} \prod_{i=1}^{q} L_i$ yields

$$\varepsilon_{\mathrm{p}} = -\frac{\pi v}{L^{d+1}} \sum_{j=q}^{d-1} \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j(q)$$

$$\varepsilon_{\mathrm{N}} = -\frac{\pi v}{(2L)^{d+1}} \sum_{j=q}^{d-1} \sum_{m=j-q+1}^{d-q} \binom{d-q}{m} \left(\Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j(q)\right) \qquad (38)$$

$$\varepsilon_{\mathrm{D}} = \frac{\pi v}{(2L)^{d+1}} \sum_{j=q}^{d-1} (-1)^{d+j} \binom{d-q-1}{j-q} \left(\Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j(q)\right).$$

The three equations in (38) are our final results for the periodic, Neumann and Dirichlet Casimir energy densities for massless scalar fields moving with wave velocity v in a d-dimensional rectangular cavity where $d-q$ sides have equal length L and q sides have much larger length. The expressions contain a dominant finite analytical part plus a fast-converging remainder $R_j(q)$ given by (29). General formulae for arbitrary lengths are obtained in appendix B.

3. Physical interpretation of Casimir energy formulae

The Casimir energy formula (30) for periodic boundary conditions and (32) and (34) for Neumann and Dirichlet conditions, respectively have a clear physical picture or interpretation. Excluding the remainder, the formulae can be viewed as the energy needed to set up the parallel plates from which the rectangular cavity is constructed. For example, consider the case $d = 3$ and $q = 0$ corresponding to a cube (hypertorus for periodic) with sides of length L. The cube is built out of three sets of parallel plates. In (30) this corresponds to summing the term $\Gamma\left(\frac{j+2}{2}\right)\pi^{\frac{-j-4}{2}}\zeta(j+2)$ for $j = 0, 1$ and 2. To build the cube, one begins by placing two plates a distance L apart. This corresponds to $j = 2$. Adding two more plates corresponds to $j = 1$ and the last two plates complete the cube and correspond to $j = 0$. We now show mathematically that the Casimir energy is the sum of parallel plate energies plus a remainder. Consider periodic boundary conditions. The energy for parallel plates defined by letting $q = d - 1$ in (30) is

$$E_{\mathrm{p}_{\parallel}}(d) = -\frac{\pi v}{L^d} \prod_{i=1}^{d-1} L_i \Gamma\left(\frac{d+1}{2}\right) \pi^{\frac{-d-3}{2}} \zeta(d+1). \qquad (39)$$

$R_j(q)$ is zero for parallel plates and this is why it is not present in (39). The parallel plate energy in $j + 1$ dimensions is

$$E_{\mathrm{p}_{\parallel}}(j+1) = -\frac{\pi v}{L^{j+1}} \prod_{i=1}^{j} L_i \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2). \qquad (40)$$

In (30), $j \geqslant q$. Therefore the first q products in $\prod_{i=1}^{j} L_i$ are large and the rest are equal to L so that the above product $\prod_{i=1}^{j} L_i$ can be replaced by $L^{j-q} \prod_{i=1}^{q} L_i$ yielding

$$E_{\mathrm{p}_{\parallel}}(j+1) = -\frac{\pi v}{L^{q+1}} \prod_{i=1}^{q} L_i \Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2). \qquad (41)$$

Substituting (41) into (30) yields:

$$E_{\mathrm{p}}(q, d) = \sum_{j=q}^{d-1} \left(E_{\mathrm{p}_{\parallel}}(j+1) + R_j(q)\right). \qquad (42)$$

As can be seen, the Casimir energy in a d-dimensional space with q large dimensions is the sum of parallel plates immersed in different dimensions plus a remainder. When building the rectangular cavity out of successive parallel plates, the first parallel plates have $d - 1$ large dimensions, the second have $d - 2$ large dimensions and so on until the last set which has q large dimensions. In short, the $(d - q)$-dimensional resonator is the sum of one-dimensional resonators each immersed in a different dimension ranging from $q + 1$ to $d - 1$.

What is the physical interpretation for the remainder? The energies for parallel plates are by definition those for isolated plates in vacuum. However, to construct the rectangular cavity, one adds plates to other plates already present. To clarify this difference consider two scenarios. Scenario I: plates are brought together in vacuum in a two-dimensional space. This leaves one dimension which is large. Scenario II: consider a three-dimensional space where there is already a pair of parallel plates. Now add another pair of plates. This leaves one dimension which is large as in scenario I. The main point is this: the energy in scenario II for adding the second set of plates is almost but not exactly equal to the energy of the plates in scenario I. The reason is that in scenario II there is also an interaction energy due to the presence of the other plates. The remainder term is therefore an 'interaction' or potential energy arising from the nonlinearity of the energy when waves moving along different directions are added. By interaction energy we do not mean that there is a Feynman diagram where scalar fields meet at a vertex. That would be a nonlinear theory like $\lambda\phi^4$. What we have here is a linear theory and the waves obey the superposition principle. However, the energy is clearly not linear. This is reminiscent of what occurs in classical electrodynamics. In vacuum, the theory is linear and one can add two electric field vectors but the energy itself is not linear since it is proportional to the square of the electric field. What we usually call the potential energy between two static charges q_1 and q_2 is nothing but the interaction energy between the electric field $\mathbf{E}_1$ produced by the first charge and the electric field $\mathbf{E}_2$ produced by the second charge. The energy density is proportional to $(\mathbf{E}_1 + \mathbf{E}_2)^2 = E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2$ and the integration of the cross term $2\mathbf{E}_1 \cdot \mathbf{E}_2$ over all space yields the well-known potential energy proportional to $q_1 q_2 / r$ where r is the distance between the charges. The remainder term is similarly a potential energy arising from the nonlinearity of the energy.

We can now make predictions about the behaviour of the remainder for periodic, Neumann and Dirichlet boundary conditions. We predict the following:

- percentage wise, the periodic case will have the highest remainder, the Dirichlet case the smallest, and Neumann in between;
- the remainder grows with the space dimension for the periodic and Neumann cases but actually decreases for the Dirichlet case.

Let us see how we can make such predictions. The Casimir energy is the difference between discrete and continuum modes. As the frequency increases the discrete approaches the continuum. Therefore the Casimir energy picks out the low-frequency or low-energy behaviour. Moreover, the lower the energy, the more nonlinear is the change in energy. Higher energies are closer to the continuum and changes are more linear. The minimum energy mode for the periodic and Neumann cases is zero (the case when all n_i's are zero). For Dirichlet the minimum energy mode occurs when all n_i's are equal to 1. For concreteness let the space dimension be 5. For periodic and Neumann the smallest nonzero energy state occurs when one n_i is 1 so that one of five slots is filled with 1, e.g. $(0, 1, 0, 0, 0)$ while for Dirichlet the minimum energy starts at $(1, 1, 1, 1, 1)$. Now add 1 to both cases (creating states with two 1's like $(0, 1, 0, 0, 1)$ and states like $(1, 2, 1, 1, 1)$). The percentage change in the energy in the Dirichlet case will not be large because the energy started off large. The energy changes almost linearly leading to a small remainder. As the dimension increases, the energy for the

Dirichlet case starts off even higher and the change is even less. For Dirichlet, we therefore predict the remainder to be a very small percentage of the energy and that it *decreases* as the space dimension grows. In the periodic and Neumann case, the energy starts off low, so the change is a larger percentage of the initial energy and therefore more nonlinear than in the Dirichlet case. This effect is greatly accentuated by the fact that are many more low-energy combinations for the Neumann and periodic case compared to the Dirichlet case. For example, there are five ways to place 2 in (1, 2, 1, 1, 1) but there are ten ways to arrange the two 1's in (0, 1, 0, 0, 1). The remainder will therefore be considerably larger in the Neumann and periodic cases. Moreover, the remainder for periodic and Neumann cases will grow as the dimension increases because as the number of zeros increases there are simply more possible low-energy combinations and this increases the nonlinear effect. Finally, the periodic case has the largest remainder of all the cases because negative n's are allowed, so that in our state (0, 1, 0, 0, 1) one can also have combinations with -1 leading to considerably more low-energy contributions than in the Neumann case. Our numerical results confirm all these trends.

4. Epstein zeta in odd dimensions as products of one-dimensional sums plus remainder

When applied to a rectangular geometry, the zeta function regularization technique via analytical continuation yields quickly a finite expression for the Casimir energy in terms of homogeneous Epstein zeta functions. The subtraction of two infinities does not explicitly appear anywhere in the process. This is a great advantage over the cut-off technique. We use zeta function regularization here to obtain quickly an expression for the Casimir energy in terms of Epstein zeta functions for the periodic case. Our main goal however is to express the homogeneous Epstein zeta function for 3, 5 and 7 dimensions in terms of products of one-dimensional sums plus a small remainder. Readers interested in getting a deeper understanding of the zeta regularization technique as well as other techniques such as heat-kernel methods are referred to the following books [55–57]. A sample of older and more recent articles where these techniques are applied in various contexts ranging from gravitation to condensed matter can be found in [58–71]. For the case of rectangular cavities in arbitrary d dimensions, these techniques have been applied in [42–44].

Though one can compute a finite numerical result, extra work must be done to express the Epstein zeta function in a compact analytical form. Define the Epstein zeta function $Z_d(a_1, \ldots, a_d; s)$ as

$$Z_d(a_1, \ldots, a_d; s) \equiv \sideset{}{'}\sum_{\substack{n_i=-\infty \\ i=1,\ldots,d}}^{\infty} [(a_1 n_1)^2 + \cdots + (a_d n_d)^2]^{-s}, \qquad (43)$$

where the prime excludes the case where all n's are zero and absolute convergence requires Re $s > d/2$. *Our definition differs from the standard one by a factor of 2 in the power, i.e. we have $-s$ instead of $-s/2$.* This definition is chosen as it simplifies our final expressions. We focus on the case of the hypercube, where all a's are equal and can be pulled out of the sum in (43) (for simplicity we set them to unity). This yields the homogeneous Epstein zeta function $Z_d(s)$. The vacuum energy in d dimensions for periodic boundary conditions is trivial to write in terms of $Z_d(s)$:

$$E_{\mathrm{p}}^{\mathrm{vac}}(0, d) = \frac{\pi v}{L} \sum_{\substack{n_i=-\infty \\ i=1,\ldots,d}}^{\infty} \left(n_1^2 + \cdots + n_d^2\right)^{1/2}$$

$$= \frac{\pi v}{L} Z_d(-1/2). \qquad (44)$$

Now $Z_d(-1/2)$ is formally infinite if (43) is applied in a straightforward fashion. It therefore requires regularization. The keystone of the zeta regularization technique is analytic continuation and the existence of a reflection formula. Like the Riemann zeta function, the Epstein zeta function has an integral representation which yields an analytic continuation over the entire complex plane except for a pole at $s = d/2$. The representation leads to the following functional relation or reflection formula:

$$\pi^{-s}\Gamma(s)Z_d(s) = \pi^{s-d/2}\Gamma(d/2 - s)Z_d(d/2 - s). \tag{45}$$

We therefore obtain that

$$Z_d(-1/2) = -0.5Z_d\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)\pi^{\frac{-3-d}{2}} \tag{46}$$

and (44) reduces to the Casimir energy

$$E_{\mathrm{p}}(0, d) = \frac{-\pi v}{2L}Z_d\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)\pi^{\frac{-3-d}{2}}. \tag{47}$$

Clearly, the Casimir energy is finite since $Z_d\left(\frac{d+1}{2}\right)$ converges. The reader should appreciate just how quickly the zeta function technique yields this result.

The homogeneous Epstein zeta function $Z_d(s)$ can be expressed in terms of sums over the arithmetical function $r_d(n)$ which is the number of representations of an integer n as a sum of d squares without regard to sign or order:

$$
\begin{aligned}
Z_d(s) &\equiv \sum_{\substack{n_i=-\infty \\ i=1,\ldots,d}}^{\infty}{}' \left[n_1^2 + \cdots + n_d^2\right]^{-s} \\
&= \sum_{n=1}^{\infty} \frac{r_d(n)}{n^s}.
\end{aligned}
\tag{48}
$$

We can therefore use results from number theory on $r_d(n)$ to obtain directly formulae for the Epstein zeta function. It turns out that formulae for $r_d(n)$ which are not complicated exist in 2, 4, 6 and 8 dimensions and these can be used to obtain the Epstein zeta function (48) as products of one-dimensional sums with no remainder. The formula for dimension 1 is trivial (by definition a Riemann zeta function) but formulae for 3, 5 and 7 dimensions eluded number theorists until a major breakthrough in 2002 when Goro Shimura developed a systematic way of finding formulae for $r_d(n)$ for values of d up to 8 [50]. Unfortunately, the odd-dimensional formulae are much more complicated than the even ones. However, one can develop a technique where one obtains excellent analytical expressions plus a small remainder for $Z_3(s)$, $Z_5(s)$ and $Z_7(s)$. This technique makes use of number theory results in 2, 4, 6 and 8 dimensions and the Euler–Maclaurin formula to fill in the odd-dimensional gaps. The remainder which is explicitly evaluated turns out small because the odd cases are derived to a large part from the even cases. The most important case is of course $Z_3\left(-\frac{1}{2}\right)$ since it relates to the realistic three-dimensional Casimir energy. We obtain a nice compact analytical expression for $Z_3(s)$. The analytical part is so accurate that it yields the correct Casimir energy to within a remarkable 0.04% as compared to 1.6% from our cut-off formulae.

We start by stating the number-theoretic formulae for $r_2(n)$, $r_4(n)$, $r_6(n)$ and $r_8(n)$ and the known exact expressions for Z_1, Z_2, Z_4, Z_6 and Z_8 obtained from them via (48). We illustrate how to obtain $Z_d(s)$ via the number-theoretic formulae for $r_d(n)$, something that may not be too familiar to many physicists. We choose $d = 6$ as the example to illustrate as it fills a gap in the table quoted in [42] which contains Z_1, Z_2, Z_4 and Z_8 but not Z_6. We then develop the mathematical technique by which we obtain the odd-dimensional homogeneous Epstein zeta functions.

4.1. Exact expressions for even-dimensional Epstein zeta function via $r_d(n)$

As mentioned already, the arithmetical function $r_d(n)$ is the number of representations of an integer n as the sum of d squares without regard to order or sign. The formulae for $r_d(n)$ for $d = 2, 4, 6$ and 8 are known and given by (a good history with references can be found in [72])

$$r_2(n) = 4 \sum_{d|n} \chi(d) \qquad\qquad r_4(n) = 8 \sum_{\substack{d|n \\ 4 \nmid d}} d$$

$$r_6(n) = 16 \sum_{d|n} \chi(d') d^2 - 4 \sum_{d|n} \chi(d) d^2 \qquad r_8(n) = 16 \sum_{d|n} (-1)^{n+d} d^3 \tag{49}$$

where $d' = n/d$ and $\chi(d)$ is the primitive Dirichlet character modulo 4 given by $\chi(d) = 0$ if d is even and $\chi(d) = (-1)^{\frac{d-1}{2}}$ if d is odd. We now evaluate $Z_d(s)$ for $d = 6$

$$
\begin{aligned}
Z_6(s) &= \sum_{n=1}^{\infty} \frac{r_6(n)}{n^s} \\
&= 16 \sum_{d'=\text{odd}} \sum_{d=1}^{\infty} \frac{(-1)^{\frac{d'-1}{2}} d^2}{(d'd)^s} - 4 \sum_{d=\text{odd}} \sum_{p=1}^{\infty} \frac{(-1)^{\frac{d-1}{2}} d^2}{(dp)^s} \\
&= 16 \sum_{m=0}^{\infty} \sum_{d=1}^{\infty} \frac{(-1)^m}{(2m+1)^s d^{s-2}} - 4 \sum_{m=0}^{\infty} \sum_{p=1}^{\infty} \frac{(-1)^m}{(2m+1)^{s-2} p^s} \\
&= 16 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s} \sum_{d=1}^{\infty} \frac{1}{d^{s-2}} - 4 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{s-2}} \sum_{p=1}^{\infty} \frac{1}{p^s} \\
&= 16\beta(s)\zeta(s-2) - 4\beta(s-2)\zeta(s), \tag{50}
\end{aligned}
$$

where $\beta(s)$ and $\zeta(s)$ are the Dirichlet beta and Riemann zeta functions, respectively defined by $\beta(s) \equiv \sum_{n=0}^{\infty} (-1)^n/(2n+1)^s$ and $\zeta(s) \equiv \sum_{n=1}^{\infty} 1/n^s$. We have illustrated how knowledge of the arithmetical function $r_6(n)$ leads to an exact and simple representation for the Epstein zeta function $Z_6(n)$ as a product of the one-dimensional sums $\beta(s)$ and $\zeta(s)$. The other Epstein zeta functions can be obtained in a similar fashion. We state them below together with $Z_6(s)$ [49]:

$$
\begin{aligned}
&Z_1(s) = 2\zeta(2s), \qquad Z_2(s) = 4\zeta(s)\beta(s) \\
&Z_4(s) = 8\zeta(s)\zeta(s-1)(1 - 4^{1-s}) \\
&Z_6(s) = 16\beta(s)\zeta(s-2) - 4\beta(s-2)\zeta(s) \\
&Z_8(s) = 16\zeta(s)\zeta(s-3)(1 - 2^{1-s} + 4^{2-s}). \tag{51}
\end{aligned}
$$

4.2. Analytical expressions for Epstein zeta function $Z_d(s)$ in 3, 5 and 7 dimensions

As already mentioned, the formulae for $r_d(n)$ for $d = 3, 5$ and 7 are much more complicated than the even ones and it is not easy to use them to obtain analytical formulae for Z_3, Z_5 and Z_7. We therefore develop a separate technique to find such expressions. The Epstein zeta function $Z_d(s)$ defined in (48) contains d sums which begin at $-\infty$. It is convenient to define another function $P_k(s)$ as k sums which start at 1:

$$P_k(s) \equiv \sum_{\substack{n_i=1 \\ i=1,\dots,k}}^{\infty} \left[n_1^2 + \cdots + n_k^2 \right]^{-s}. \tag{52}$$

We can express $P_k(s)$ as sums over $Z_m(s)$:

$$P_k(s) = \sum_{m=1}^{k} (-1)^{m+k} 2^{-k} \binom{k}{m} Z_m(s). \tag{53}$$

Similarly, we can express $Z_d(s)$ as sums over $P_k(s)$:

$$Z_d(s) = \sum_{k=1}^{d} \binom{d}{k} 2^k P_k(s). \tag{54}$$

It is instructive to map out the main idea or process behind the technique we will use. Consider the example of wanting to find expressions for Z_3. From (54), you would need to know P_1, P_2 and P_3. You can find P_1 and P_2 from (53) since analytical expressions for Z_1 and Z_2 are known. However, you do not know P_3. At this point, you use the Euler–Maclaurin formula to express P_3 in terms of P_2 plus a remainder. Again, you know P_2 in terms of Z_1 and Z_2, so that you can finally express Z_3 in terms of Z_1, Z_2 and a remainder and hence as an analytical part plus a remainder. The process can be continued to find expressions for Z_5 and Z_7 (and even Z_9 if one wants to but the expression becomes cumbersome). We now develop the mathematical technique and obtain our main equation. $Z_d(s)$ given by (54) can be expanded as

$$Z_d(s) = \sum_{k=1}^{d-1} \binom{d}{k} 2^k P_k(s) + 2^d P_d(s)$$

$$= \sum_{k=1}^{d-1} (-1)^{d+k+1} \binom{d}{k} Z_k(s) + 2^d P_d(s), \tag{55}$$

where (53) was used. We now express $P_d(s)$ in terms of $P_{d-1}(s)$ plus a remainder via the Euler–Maclaurin formula (9):

$$P_d(s) = \sum_{\substack{n_i=1 \\ i=1,\ldots,d}}^{\infty} \left[n_1^2 + \cdots + n_d^2 \right]^{-s} = \sum_{\substack{n_i=1 \\ i=1,\ldots,d-1}}^{\infty} \int_0^{\infty} \frac{dx}{(x^2 + n^2)^s} - \frac{1}{2n^{2s}}$$

$$- \frac{1}{2} \sum_{\nu=0}^{\infty} \int_0^1 B_2(x) \frac{\partial^2}{\partial x^2} \frac{1}{((x+\nu)^2 + n^2)^s} \, dx, \tag{56}$$

where

$$n^2 \equiv n_1^2 + \cdots + n_{d-1}^2. \tag{57}$$

The first integral in (56) can readily be evaluated:

$$\int_0^{\infty} \frac{dx}{(x^2 + n^2)^s} = \frac{1}{n^{2s-1}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{\sqrt{\pi}}{2} = \frac{\alpha(s)}{2} \frac{1}{n^{2s-1}}, \tag{58}$$

where $\alpha(s)$ is defined by

$$\alpha(s) \equiv \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}. \tag{59}$$

Inserting (58) into (56) yields

$$P_d(s) = \sum_{\substack{n_i=1 \\ i=1,\ldots,d-1}}^{\infty} \frac{\alpha(s)}{2n^{2s-1}} - \frac{1}{2n^{2s}} - \frac{1}{2} \sum_{\nu=0}^{\infty} \int_0^1 B_2(x) \frac{\partial^2}{\partial x^2} \frac{1}{((x+\nu)^2 + n^2)^s} \, dx. \tag{60}$$

By definition

$$\sum_{\substack{n_i=1 \\ i=1,\ldots,d-1}}^{\infty} \frac{1}{n^{2s}} = P_{d-1}(s).$$

Therefore

$$P_d(s) = \frac{\alpha(s)}{2} P_{d-1}\left(s - \frac{1}{2}\right) - \frac{1}{2} P_{d-1}(s) + R_d(s) \qquad (61)$$

where $R_d(s)$ is the remainder defined by

$$R_d(s) \equiv \sum_{n_1,\ldots,n_{d-1}=1}^{\infty} \frac{-1}{2} \sum_{v=0}^{\infty} \int_0^1 B_2(x) \frac{\partial^2}{\partial x^2} \frac{1}{((x+v)^2 + n^2)^s} \, dx. \qquad (62)$$

The remainder $R_d(s)$ is worked out in appendix C and the result is

$$R_d(s) = \sum_{n_1,\ldots,n_{d-1}=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{2}{\sqrt{\pi}} \left(\frac{\pi \ell}{n}\right)^{s-1/2} \Gamma(1-s) \sin(\pi s) K_{s-1/2}(2\pi \ell n), \qquad (63)$$

where n is given by (57). We now evaluate the term $2^d P_d(s)$ occurring in (55) via (61) and (53):

$$2^d P_d(s) = 2^{d-1}\alpha(s) P_{d-1}\left(s - \tfrac{1}{2}\right) - 2^{d-1} P_{d-1}(s) + 2^d R_d(s)$$
$$= \sum_{m=1}^{d-1} (-1)^{d+m-1} \binom{d-1}{m} \left[\alpha(s) Z_m\left(s - \tfrac{1}{2}\right) - Z_m(s)\right] + 2^d R_d(s). \qquad (64)$$

Substituting (64) into (55) we obtain our main equation:

$$Z_d(s) = \sum_{m=1}^{d-1} (-1)^{d+m-1} \left[\alpha(s) \binom{d-1}{m} Z_m\left(s - \tfrac{1}{2}\right) + \binom{d-1}{m-1} Z_m(s)\right] + 2^d R_d(s), \qquad (65)$$

where $R_d(s)$ is the remainder given by (63). Equation (65) expresses Z_d as sums over Z_i's from 1 to $d-1$ plus a remainder. We are now in a position to obtain expressions for Z_3, Z_5 and Z_7 as products of one-dimensional sums plus a remainder by using our main equation (65) together with the analytical expressions for Z_1, Z_2, Z_4, Z_6 and Z_8 given in (51). We begin with $Z_3(s)$. Applying equation (65) yields

$$Z_3(s) = \alpha(s)\left[-2Z_1\left(s - \tfrac{1}{2}\right) + Z_2\left(s - \tfrac{1}{2}\right)\right] - Z_1(s) + 2Z_2(s) + 2^3 R_3(s). \qquad (66)$$

We now substitute the analytical expressions for Z_1 and Z_2 given in (51) and obtain our final expression for Z_3:

$$Z_3(s) = 4\alpha(s)\zeta\left(s - \tfrac{1}{2}\right)\beta\left(s - \tfrac{1}{2}\right) - 4\alpha(s)\zeta(2s-1) + 8\zeta(s)\beta(s) - 2\zeta(2s) + 8R_3(s). \qquad (67)$$

This is a compact analytical result for the important three-dimensional case. The only remainder is $8R_3(s)$ and the rest includes four analytical terms, each expressed in terms of simple one-dimensional sums and gamma functions. Later we will see that the analytical part yields numerically the correct Casimir energy to within 0.04%. We now evaluate $Z_5(s)$. Using again the main equation (65) we obtain

$$Z_5(s) = \alpha(s)\left[-4Z_1\left(s - \tfrac{1}{2}\right) + 6Z_2\left(s - \tfrac{1}{2}\right) - 4Z_3\left(s - \tfrac{1}{2}\right) + Z_4\left(s - \tfrac{1}{2}\right)\right]$$
$$- Z_1(s) + 4Z_2(s) - 6Z_3(s) + 4Z_4(s) + 2^5 R_5(s). \qquad (68)$$

Substituting the analytical expressions for Z_1, Z_2 and Z_4 given in (51) and Z_3 from (67) into (68) one obtains the final expression for $Z_5(s)$:

$$
\begin{aligned}
Z_5(s) = {} & 10\zeta(2s) - 32\zeta(s)\beta(s) + 32\zeta(s)\zeta(s-1)(1-4^{1-s}) \\
& + 8\alpha(s)\left[3\zeta(2s-1) - 4\zeta\left(s-\tfrac{1}{2}\right)\beta\left(s-\tfrac{1}{2}\right) + \zeta(s)\zeta\left(s-\tfrac{3}{2}\right)(1-2^{3-2s})\right] \\
& - 16\alpha(s)\alpha\left(s-\tfrac{1}{2}\right)\left(\zeta(s-1)\beta(s-1) - \zeta(2s-2)\right) + RZ_5(s),
\end{aligned}
\tag{69}
$$

where the remainder $RZ_5(s) = -32\alpha(s)R_3\left(s-\tfrac{1}{2}\right) - 48R_3(s) + 32R_5(s)$. The expression for $Z_7(s)$ is

$$
\begin{aligned}
Z_7(s) = {} & \alpha(s)\left[-6Z_1\left(s-\tfrac{1}{2}\right) + 15Z_2\left(s-\tfrac{1}{2}\right) - 20Z_3\left(s-\tfrac{1}{2}\right)\right. \\
& \left. + 15Z_4\left(s-\tfrac{1}{2}\right) - 6Z_5\left(s-\tfrac{1}{2}\right) + Z_6\left(s-\tfrac{1}{2}\right)\right] - Z_1(s) + 6Z_2(s) \\
& - 15Z_3(s) + 20Z_4(s) - 15Z_5(s) + 6Z_6(s) + 2^7 R_7(s)
\end{aligned}
\tag{70}
$$

where Z_1, Z_2, Z_4 and Z_6 are given by (51), Z_3 by (67) and Z_5 by (69). It would be cumbersome to write out the analytical terms for Z_7 as we did for Z_3 and Z_5. For calculations, one simply evaluates the necessary Z's and substitutes them in (70). This ends our results for the odd-dimensional Epstein zeta functions. One could have continued and obtained expressions for $Z_9(s)$ but this is no longer interesting as the expressions become way too long. We now state and discuss the numerical results for the Casimir energy.

5. Numerical results and discussion

Table 1 contains the numerical results for the Casimir energy density for periodic (ε_p), Dirichlet (ε_D) and Neumann (ε_N) for q large dimensions and $d - q$ dimensions of equal length L. This is calculated using the formulae in (38) and equation (29) for the remainder $R_j(q)$ (v and L are assumed to be unity). We state the analytical and remainder contribution separately and calculate their sum to obtain the Casimir energy density. For dimensions up to $d = 5$, we include all values of q. For higher dimensions up to $d = 10$ we only state $q = 0$. For numerical results in the case where one has arbitrary lengths the reader is referred to [43–45]. The formulae derived in appendix B are actually very well suited for such a numerical study but length limitations restrict us here.

Table 1 shows that the absolute value of the Casimir energy density for the periodic case is the largest, followed by the Neumann and Dirichlet. Note that the sign in the Dirichlet case alternates in two fashions: for a given q, it alternates as the dimension d changes and it also alternates as q changes for a given d. The Casimir energy densities agree with a few exceptions with results obtained by computing the Epstein zeta function and quoted in the table in [42]. For periodic boundary conditions, results for $d = p$ (corresponding to $q = 0$ in our case) are close to our values but do not fully agree. For $d = 2$ the values agree but for $d = 3$ they obtain -0.81 while we obtain -0.838. For $d = 4$, they obtain -0.85 while we obtain -0.932 and for $d = 5$ they obtain -0.95 while we obtain -1.022. Which values are correct? Table 3 contains an independent determination of the Casimir energy density for the case $q = 0$ for periodic boundary conditions. The values in table 3 for $d = 3, 4$ and 5 are $-0.837\,537$, $-0.932\,077$ and $-1.022\,83$, respectively, and these values are in agreement with our results. Therefore, in the few places where our results differ from [42], our numerical values can be considered correct. Some numerical results are also quoted for Dirichlet boundary conditions in [43, 44] where Epstein zeta functions were also used. In [43], the column $u = 0$ corresponds to our $q = 0$ and are in agreement. In [44] where D is the spacetime dimension, i.e. $D = d + 1$, their first column corresponds to our $d - q = 2$ results and are in agreement.

Table 1. Analytical and remainder contributions to Casimir energy density for periodic, Dirichlet and Neumann boundary conditions.

(q, d)	Periodic			Dirichlet			Neumann		
	ε_{p}	Analytical	Remainder	ε_{D}	Analytical	Remainder	ε_{N}	Analytical	Remainder
0, 2	−0.718 873	−0.714 9121	−0.003 9607	0.041 041	0.041 5357	−0.000 4951	−0.220 759	−0.220 2637	−0.000 4951
1, 2	−0.191 313	−0.191 3133	0	−0.023 914	−0.023 9142	0	−0.023 914	−0.023 9142	0
0, 3	−0.837 537	−0.824 5743	−0.012 9625	−0.015 732	−0.015 6650	−0.000 0675	−0.285 309	−0.283 7567	−0.001 5528
1, 3	−0.305 322	−0.300 9756	−0.004 3463	0.004 832	0.005 1032	−0.002 7164	−0.042 997	−0.042 7251	−0.000 2716
2, 3	−0.109 662	−0.109 6623	0	−0.006 854	−0.006 8539	0	−0.006 854	−0.006 8539	0
0, 4	−0.932 077	−0.903 3714	−0.028 7054	0.006 226	0.006 2453	−0.000 0194	−0.334 058	−0.330 7980	−0.003 2600
1, 4	−0.394 299	−0.379 7726	−0.014 5260	−0.001 634	−0.001 5871	−0.000 0465	−0.058 881	−0.058 0200	−0.000 8614
2, 4	−0.193 407	−0.188 4593	−0.004 9473	0.000 810	0.000 9645	−0.000 1546	−0.012 898	−0.012 7432	−0.000 1546
3, 4	−0.078 797	−0.078 7971	0	−0.002 462	−0.002 4624	0	−0.002 462	−0.002 4624	0
0, 5	−1.022 83	−0.968 9932	−0.053 8395	−0.002 611	−0.002 6055	−0.000 0052	−0.372 895	−0.367 1673	−0.005 7281
1, 5	−0.478 283	−0.445 3944	−0.032 8889	0.000 504	0.000 5171	−0.000 0135	−0.072 698	−0.070 8686	−0.001 8292
2, 5	−0.270 975	−0.254 0811	−0.016 8939	−0.000 308	−0.000 2764	−0.000 0321	−0.018 440	−0.001 7945	−0.000 4959
3, 5	−0.150 257	−0.144 4188	−0.005 8383	0.000 115	0.000 2059	−0.000 0912	−.000 4810	−0.004 7190	−0.000 0912
4, 5	−0.065 622	−0.065 6218	0	−0.001 025	−0.001 0253	0	−0.001 025	−0.001 0253	0
0, 6	−1.122 49	−1.029 970	−0.092 517	0.001 114	0.001 1158	−0.000 0017	−0.405 594	−0.396 4942	−0.009 1000
0, 7	−1.243 13	−1.091 817	−0.151 318	−0.000 489	−0.000 4884	−0.000 0006	−0.434 680	−0.421 1207	−0.013 5591
0, 8	−1.400 15	−1.159 323	−0.240 830	0.000 217	0.000 2170	−0.000 0002	−0.461 950	−0.442 6076	−0.019 3419
0, 9	−1.616 21	−1.237 827	−0.378 385	−0.000 098	−0.000 0977	−0.000 0001	−0.488 792	−0.462 0371	−0.026 7547
0, 10	−1.927 25	−1.334 376	−0.592 8761	0.000 044	0.000 0444	0.000 0000	−0.516 394	−0.480 1973	−0.036 19642

Table 2. Percentage of Casimir energy which is remainder (case $q = 0$).

d	Dirichlet	Neumann	Periodic
2	1.2	0.22	0.55
3	0.42	0.54	1.6
4	0.31	0.98	3.1
5	0.20	1.5	5.3
6	0.15	2.2	8.2
7	0.12	3.1	12
8	0.09	4.2	17
9	0.10	5.5	23
10	0.00	7.0	31

Table 3. Epstein-zeta function and comparison of remainder with cut-off.

	Casimir energy	Epstein zeta		% Remainder	
d ($q = 0$)	density (periodic)	Analytical	Remainder	Epstein zeta	Cut-off
2	−0.718 873	−0.718 873	0	0	0.6
3	−0.837 537	−0.837 2276	−0.000 3928	0.04	1.6
4	−0.932 077	−0.932 077	0	0	3.1
5	−1.022 83	−1.025 582	0.002 7514	0.3	5.3
6	−1.122 49	−1.122 49	0	0	8.2
7	−1.243 13	−1.197 224	−0.045 9060	3.7	12.2
8	−1.400 15	−1.400 15	0	0	17.2

In table 2, the percentage of the Casimir energy which is a remainder is quoted for the different boundary conditions as a function of the dimension d (for simplicity, we quote the hypercube case $q = 0$ but the same trend is followed by all q values). Table 2 confirms the predictions made in section 3. Moving down the table, as the dimension increases, the percentage decreases for Dirichlet but increases for Neumann and periodic as predicted in section 3. Moving horizontally across the table the percentage is lowest for Dirichlet and largest for periodic with Neumann in between, again as predicted in section 3 (with the only exception being $d = 2$ due to the limited low-energy permutations in the periodic and Neumann cases and the fact that the Dirichlet starts off at a low energy unlike higher dimensions).

Note how small is the percentage remainder. Only at the highest dimensions is the percentage high and this is mostly for the periodic case. The percentage remainder is negligible for the Dirichlet case and the analytical formulae are all we need. The Neumann case has a very low remainder at low dimensions. At $d = 4$ it has less than a 1% remainder so that the analytical formulae are simply excellent at lower dimensions. Even the periodic case at $d = 3$ has only a 1.6% remainder but the remainder grows rapidly with dimension compared to the other two cases.

Table 3 contains the Casimir energy for the periodic case at $q = 0$ for values of d ranging from 2 to 8 calculated via the expressions for the homogeneous Epstein zeta function $Z_d(s)$ (again v and L are assumed to be unity). Our aim here was not to make a complete table of Casimir values using the Epstein zeta function. This has already been successfully done in [42]. The goal was mainly to calculate the analytical and remainder terms for the homogeneous Epstein zeta function in 3, 5 and 7 dimensions. For even dimensions, the expressions are calculated via (51) where there is no remainder. For the odd cases of 3, 5 and 7 dimensions

they are calculated via our derived expressions (67), (69) and (70), and (63) for the remainder $R_d(s)$. Note how close are the derived Epstein zeta analytical results to the actual Casimir energy and hence the small remainder percentage wise. The analytical expressions (67), (69) and (70) we derived for the Epstein zeta are limited to a few dimensions but are exceptionally accurate. As already stated, for the realistic three-dimensional case, the remainder is only a remarkable 0.04% of the Casimir energy. As one can see, the remainder for these few cases is smaller than the remainder from our cut-off technique. The reason is due to the fact that the odd-dimensional cases are derived from the even ones which contain no remainder.

Acknowledgments

Appendix A. Remainder term $R_j(q)$

In this appendix we evaluate the remainder term $R_j(q)$ defined by

$$R_j(q) \equiv \lim_{\lambda \to 0} \partial_\lambda R_j(q, \lambda), \tag{A.1}$$

where $R_j(q, \lambda)$ is given by (17), i.e.

$$R_j(q, \lambda) = \sum_{m=1}^{j-q} \sum_{n=1}^{\infty} (-1)^m \binom{j-q}{m} 2^{j-m+1} \int_0^{\infty} \int_0^1 \prod_{i=1}^m \sum_{v_i=0}^{\infty} B_2(x_i) \frac{\partial^2}{\partial x_i}$$
$$\times \exp\left(-\lambda \sqrt{n^2 + (x_1 + v_1)^2 + \cdots + (x_m + v_m)^2 + y_1^2 + \cdots + y_{j-m}^2}\right)$$
$$\times dx_1 \cdots dx_m \, dy_1 \cdots dy_{j-m}. \tag{A.2}$$

There are m integrals from 0 to 1 over the x's and $j - m$ integrals from 0 to ∞ over the y's. Our goal is to simplify (A.2) as much as possible and put it in a compact form useful for computations. At the end, the result is that (A.2) can conveniently be reduced to sums over Bessel functions. The first step is to convert the multiple integrals over the y's to a single integral by using spherical coordinates:

$$r^2 = y_1^2 + \cdots + y_{j-m}^2; \quad dy_1 \cdots dy_{j-m} = 2^{m-j+1} \frac{\pi^{\frac{j-m}{2}}}{\Gamma\left(\frac{j-m}{2}\right)} r^{j-m-1} \, dr. \tag{A.3}$$

$R_j(q, \lambda)$ is then reduced to

$$R_j(q, \lambda) = \sum_{m=1}^{j-q} \sum_{n=1}^{\infty} (-1)^m 4 \binom{j-q}{m} \frac{\pi^{\frac{j-m}{2}}}{\Gamma\left(\frac{j-m}{2}\right)} \int_0^{\infty} \int_0^1 \prod_{i=1}^m \sum_{v_i=0}^{\infty} B_2(x_i) \frac{\partial^2}{\partial x_i}$$
$$\times \exp\left(-\lambda \sqrt{n^2 + (x_1 + v_1)^2 + \cdots + (x_m + v_m)^2 + r^2}\right) dx_1 \cdots dx_m r^{j-m-1} \, dr. \tag{A.4}$$

We now turn to the x integrals from 0 to 1. Note that $x + v$ is continuous and runs from 0 to ∞. It is therefore convenient to drop the sum over v, replace $x + v$ by x and integrate from 0 to ∞ instead of 0 to 1. This is valid as long as the Bernoulli function $B_2(x)$ is replaced by $B_2(x - [x])$ where $[x]$ is the greatest integer less than or equal to x. This ensures that the Bernoulli function is periodic with period 1 while x runs to infinity. Moreover, $B_2(0) = B_2(1)$ so that $B_2(x - [x])$

is not only periodic but continuous. A Fourier expansion of $B_2(x) = x^2 - x + 1/6$ can readily be obtained and is given by

$$x^2 - x + 1/6 = \sum_{\ell=1}^{\infty} \frac{\cos(2\pi \ell x)}{\ell^2 \pi^2}. \tag{A.5}$$

The right-hand side of (A.5) is a continuous periodic function valid for all x. It is equal to the left-hand side only for $0 \leqslant x \leqslant 1$ but equal to $B_2(x - [x])$ over the entire region of integration $0 \leqslant x < \infty$. We can therefore make the following replacement:

$$\sum_{v_i=0}^{\infty} \int_0^1 B_2(x_i) \frac{\partial^2 f(x_i + v_i)}{\partial x_i} \, dx_i \rightarrow \sum_{\ell_i=1}^{\infty} \frac{1}{\ell_i^2 \pi^2} \int_0^{\infty} \cos(2\pi \ell_i x_i) \frac{\partial^2 f(x_i)}{\partial x_i} \, dx_i \tag{A.6}$$

where $f(x_i)$ is the exponential function in (A.4) with v omitted, i.e.

$$f(x_i) = \exp\left(-\lambda\sqrt{n^2 + x_1^2 + \cdots + x_i^2 + \cdots + x_m^2 + r^2}\right). \tag{A.7}$$

The function f has the following properties:

$$\lim_{x_i \to 0} \frac{\partial f(x_i)}{\partial x_i} = 0, \qquad \lim_{x_i \to \infty} \frac{\partial f(x_i)}{\partial x_i} = 0, \qquad \lim_{x_i \to \infty} f(x_i) = 0. \tag{A.8}$$

After integrating by parts twice and using the above properties of f, (A.6) reduces to

$$\int_0^{\infty} \cos(2\pi \ell_i x_i) \frac{\partial^2 f(x_i)}{\partial x_i} \, dx_i = -4\pi^2 \ell_i^2 \int_0^{\infty} \cos(2\pi \ell_i x_i) f(x_i)$$

and

$$\sum_{v=0}^{\infty} \int_0^1 B_2(x) \frac{\partial^2 f(x + v)}{\partial x} \, dx \rightarrow -4 \sum_{\ell=1}^{\infty} \int_0^{\infty} \cos(2\pi \ell x) f(x) \, dx. \tag{A.9}$$

Substituting (A.9) into equation (A.4) yields

$$R_j(q, \lambda) = \sum_{m=1}^{j-q} \sum_{n=1}^{\infty} \sum_{\ell_1,\dots,m=1}^{\infty} 4^{m+1} \binom{j-q}{m} \frac{\pi^{\frac{j-m}{2}}}{\Gamma\left(\frac{j-m}{2}\right)} \int_0^{\infty} \prod_{i=1}^{m} \cos(2\pi \ell_i x_i)$$

$$\times \exp\left(-\lambda\sqrt{n^2 + x_1^2 + \cdots + x_m^2 + r^2}\right) dx_1 \cdots dx_m r^{j-m-1} \, dr. \tag{A.10}$$

We can reduce expression (A.10) to sums over the modified Bessel function $K_{\frac{j+1}{2}}$ by applying sequentially the following set of three integrals [52]:

(I) $\quad \displaystyle\int_0^{\infty} \cos(\gamma x) \, e^{-\lambda\sqrt{b^2 + x^2}} \, dx = \frac{\lambda b}{\sqrt{\lambda^2 + \gamma^2}} K_{-1}\left(b\sqrt{\lambda^2 + \gamma^2}\right)$

(II) $\quad \displaystyle\int_0^{\infty} (x^2 + b^2)^{\mp \frac{1}{2}v} K_v\left(a\sqrt{x^2 + b^2}\right) \cos(cx) = \left(\frac{\pi}{2}\right)^{1/2} a^{\mp v} b^{\frac{1}{2}\mp v} (a^2 + c^2)^{\pm \frac{1}{2}v - \frac{1}{4}}$

$\quad\quad\quad K_{\pm v - \frac{1}{2}}\left(b\sqrt{a^2 + c^2}\right).$

(III) $\quad \displaystyle\int_0^{\infty} K_v\left(\alpha\sqrt{z^2 + x^2}\right) \frac{x^{2\mu+1}}{(z^2 + x^2)^{v/2}} \, dx = \frac{2^\mu \Gamma(\mu + 1)}{\alpha^{\mu+1} z^{v-\mu-1}} K_{v-\mu-1}(\alpha z).$

Integral I is applied once and converts the exponential and one cosine into the modified Bessel function K_{-1}, i.e.

$$\int_0^{\infty} \cos(2\pi \ell_1 x_1) \, e^{-\lambda\sqrt{b^2 + x_1^2}} \, dx_1 = \frac{\lambda b}{\sqrt{\lambda^2 + 4\pi^2 \ell_1^2}} K_{-1}\left(b\sqrt{\lambda^2 + 4\pi^2 \ell_1^2}\right) \tag{A.11}$$

where $b \equiv \sqrt{x_2^2 + \cdots + x_m^2 + n^2 + r^2}$. We now make repeated application of integral II for the remaining x's that appear in the definition of b. The subscript of the Bessel function is therefore decreased by $1/2$ each time. Since there are $m - 1$ x-integrals to perform, and we start with K_{-1}, this yields the Bessel function $K_{\frac{-m-1}{2}}$, i.e.

$$\int_0^\infty \frac{\lambda}{\sqrt{\lambda^2 + 4\pi^2 \ell_1^2}} \prod_{i=2}^m \cos(2\pi \ell_i x_i) b K_{-1}\left(b\sqrt{\lambda^2 + 4\pi^2 \ell_1^2}\right) dx_2, \ldots, dx_m$$

$$= \frac{\lambda(n^2 + r^2)^{\frac{m+1}{4}}}{\pi 2^m \left(\frac{\lambda^2}{4\pi^2} + \ell_1^2 + \cdots + \ell_m^2\right)^{\frac{m+1}{4}}} K_{\frac{-m-1}{2}}\left(2\pi\sqrt{n^2 + r^2}\sqrt{\frac{\lambda^2}{4\pi^2} + \ell_1^2 + \cdots + \ell_m^2}\right) \tag{A.12}$$

We now apply integral III to perform the integration over r, i.e.

$$\int_0^\infty \frac{\lambda \pi^{-1} 2^{-m} (n^2 + r^2)^{\frac{m+1}{4}}}{\left(\frac{\lambda^2}{4\pi^2} + \ell_1^2 + \cdots + \ell_m^2\right)^{\frac{m+1}{4}}} K_{\frac{-m-1}{2}}\left(2\pi\sqrt{n^2 + r^2}\sqrt{\frac{\lambda^2}{4\pi^2} + \ell_1^2 + \cdots + \ell_m^2}\right) r^{j-m-1} dr$$

$$= \frac{\lambda}{\pi 2^{m+1}} \frac{\Gamma\left(\frac{j-m}{2}\right)}{\pi^{\frac{j-m}{2}}} \frac{n^{\frac{j+1}{2}} K_{\frac{j+1}{2}}\left(2\pi n \sqrt{\frac{\lambda^2}{4\pi^2} + \ell_1^2 + \cdots + \ell_m^2}\right)}{\left(\frac{\lambda^2}{4\pi^2} + \ell_1^2 + \cdots + \ell_m^2\right)^{\frac{j+1}{4}}}. \tag{A.13}$$

The integrals over x and r appearing in (A.10) can now be replaced by (A.13) yielding

$$R_j(q, \lambda) = \frac{\lambda}{\pi} \sum_{m=1}^{j-q} 2^{m+1} \binom{j-q}{m} \sum_{n=1}^\infty \sum_{\ell_1, \ldots, m = 1}^\infty \frac{n^{\frac{j+1}{2}} K_{\frac{j+1}{2}}\left(2\pi n \sqrt{\frac{\lambda^2}{4\pi^2} + \ell_1^2 + \cdots + \ell_m^2}\right)}{\left(\frac{\lambda^2}{4\pi^2} + \ell_1^2 + \cdots + \ell_m^2\right)^{\frac{j+1}{4}}}. \tag{A.14}$$

Finally, by taking the derivative of $R_j(q, \lambda)$ with respect to λ and taking the limit as $\lambda \to 0$ yields our desired final result for the remainder $R_j(q)$:

$$R_j(q) \equiv \lim_{\lambda \to 0} \partial_\lambda R_j(q, \lambda)$$

$$= \frac{1}{\pi} \sum_{m=1}^{j-q} 2^{m+1} \binom{j-q}{m} \sum_{n=1}^\infty \sum_{\ell_1, \ldots, m = 1}^\infty \frac{n^{\frac{j+1}{2}} K_{\frac{j+1}{2}}\left(2\pi n \sqrt{\ell_1^2 + \cdots + \ell_m^2}\right)}{\left(\ell_1^2 + \cdots + \ell_m^2\right)^{\frac{j+1}{4}}}. \tag{A.15}$$

Our final expression (A.15) for $R_j(q)$ is excellent for numerical calculations because it converges very quickly (exponentially fast). The sums to infinity are formalities as one can reach an accuracy of eight to ten digits by summing fewer than nine numbers in each sum for j up to 10.

Appendix B. Casimir energy in rectangular cavities with arbitrary lengths

One can generalize the multidimensional cut-off method used in section 2 to obtain Casimir energy formulae for arbitrary lengths in a d-dimensional rectangular cavity. Our analysis will naturally be brief since it follows closely that of section 2 and many results from that section can be applied here. The best way to read this appendix is therefore to have section 2 and appendix A in hand for immediate reference.

The quantized frequencies ω for periodic (p), Neumann (N) and Dirichlet (D) conditions are now given by

$$\omega_p = 2\pi v \left(\frac{n_1^2}{L_1^2} + \cdots + \frac{n_d^2}{L_d^2}\right)^{1/2}, \qquad \omega_{N,D} = \pi v \left(\frac{n_1^2}{L_1^2} + \cdots + \frac{n_d^2}{L_d^2}\right)^{1/2}, \tag{B.1}$$

where the lengths range from L_1 to L_d. The regularized vacuum energy for periodic boundary conditions is then given by a similar form to (8), i.e.

$$E_{\mathrm{p}}^{\mathrm{reg}}(\lambda) = -\pi v \partial_\lambda \sum_{\substack{n_i=-\infty \\ i=1,\ldots,d}}^{\infty} e^{-\lambda\sqrt{\frac{n_1^2}{L_1^2}+\cdots+\frac{n_d^2}{L_d^2}}} = -\pi v \partial_\lambda \left(1 + \sum_{n_1=-\infty}^{\infty}{}' e^{-\lambda\sqrt{\frac{n_1^2}{L_1^2}}} + \sum_{n_2=-\infty}^{\infty}{}' \sum_{n_1=-\infty}^{\infty}\right.$$

$$\times e^{-\lambda\sqrt{\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}}} + \cdots + \left. \sum_{n_d=-\infty}^{\infty}{}' \sum_{\substack{n_i=-\infty \\ i=1,\ldots,d-1}}^{\infty} e^{-\lambda\sqrt{\frac{n_1^2}{L_1^2}+\cdots+\frac{n_d^2}{L_d^2}}}\right)$$

$$= -\pi v \sum_{j=0}^{d-1} \partial_\lambda \Lambda_j(\lambda), \tag{B.2}$$

where

$$\Lambda_j(\lambda) \equiv \sum_{n=-\infty}^{\infty}{}' \sum_{\substack{n_i=-\infty \\ i=1,\ldots,j}}^{\infty} e^{-\lambda\sqrt{\frac{n^2}{L_{j+1}^2}+\frac{n_1^2}{L_1^2}+\cdots+\frac{n_j^2}{L_j^2}}}. \tag{B.3}$$

As in (12), we obtain via the Euler–Maclaurin formula that

$$\sum_{n_i=-\infty}^{\infty} f(n_i) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - R. \tag{B.4}$$

R is given by expression (A.9) obtained in appendix A:

$$R = \sum_{v=0}^{\infty} \int_0^1 B_2(x) \frac{\partial^2 f(x+v)}{\partial x}\,\mathrm{d}x$$

$$= -4 \sum_{\ell=1}^{\infty} \int_0^{\infty} \cos(2\pi\ell x) f(x)\,\mathrm{d}x$$

$$= -2 \sum_{\ell=-\infty}^{\infty}{}' \int_0^{\infty} \cos(2\pi\ell x) f(x)\,\mathrm{d}x, \tag{B.5}$$

where we used $f(x) = f(-x)$ for the function we are considering. Then (B.4) reduces to

$$\sum_{n_i=-\infty}^{\infty} f(n_i) = 2 \sum_{\ell=-\infty}^{\infty} \int_0^{\infty} \cos(2\pi\ell x) f(x)\,\mathrm{d}x, \tag{B.6}$$

where $\ell = 0$ is now included. Therefore the j-dimensional sum appearing in (B.3) for $\Lambda_j(\lambda)$ can be obtained by repeated application of (B.6). What appears in the regularized energy (B.2) is the derivative $\partial_\lambda \Lambda_j(\lambda)$:

$$\partial_\lambda \Lambda_j(\lambda) = \partial_\lambda \sum_{n=-\infty}^{\infty}{}' \sum_{\substack{n_i=-\infty \\ i=1,\ldots,j}}^{\infty} e^{-\lambda\sqrt{\frac{n^2}{L_{j+1}^2}+\frac{n_1^2}{L_1^2}+\cdots+\frac{n_j^2}{L_j^2}}}$$

$$= \partial_\lambda \sum_{n=-\infty}^{\infty}{}' 2^j \sum_{\substack{l_i=-\infty \\ i=1,\ldots,j}}^{\infty} \int_0^{\infty} \cos(2\pi\ell_1 x_1)\ldots\cos(2\pi\ell_j x_j)\, e^{-\lambda\sqrt{\frac{n^2}{L_{j+1}^2}+\frac{x_1^2}{L_1^2}+\cdots+\frac{x_j^2}{L_j^2}}}\,\mathrm{d}x_1\ldots\mathrm{d}x_j$$

$$= \frac{L_1\ldots L_j}{(L_{j+1})^{j+1}} \left(2^{j+1}\partial_\lambda \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\lambda\sqrt{n^2+x_1^2+\cdots+x_j^2}}\,\mathrm{d}x_1\ldots\mathrm{d}x_j + \partial_\lambda R_j(\lambda)\right), \tag{B.7}$$

where the sum over all ℓ's was divided into two cases leading to the two terms in the brackets of (B.7). The first term occurs when all ℓ's are equal to zero. The second term is for all other ℓ's and corresponds to the remainder :

$$\partial_\lambda R_j(\lambda) \equiv 2^{j+1} \partial_\lambda \sum_{n=1}^{\infty} \underset{\substack{l_i=-\infty \\ i=1,\ldots,j}}{\sum_{}^{\infty}}{}' \int_0^\infty \cos\left(2\pi\ell_1 \frac{L_1}{L_{j+1}} x_1\right)$$

$$\times \cdots \cos\left(2\pi\ell_j \frac{L_j}{L_{j+1}} x_j\right) e^{-\lambda\sqrt{n^2+x_1^2+\cdots+x_j^2}} \, dx_1 \cdots dx_j, \tag{B.8}$$

where the prime over the multiple sum excludes only the case when all ℓ's are equal to zero. The multiple integral over j cosines can be obtained directly from (A.12) in appendix A by the following substitutions: $m \to j$, $\ell_i \to \ell_i L_i / L_{j+1}$ and $n^2 + r^2 \to n^2$, i.e.

$$\partial_\lambda R_j(\lambda) = \partial_\lambda \sum_{n=1}^{\infty} \underset{\substack{l_i=-\infty \\ i=1,\ldots,j}}{\sum_{}^{\infty}}{}' \frac{2\lambda(nL_{j+1})^{\frac{j+1}{2}} K_{\frac{j+1}{2}}\left(\frac{2\pi n}{L_{j+1}} \sqrt{\frac{(\lambda L_{j+1})^2}{4\pi^2} + (\ell_1 L_1)^2 + \cdots + \left(\ell_j^2 L_j\right)^2}\right)}{\pi \left(\frac{(\lambda L_{j+1})^2}{4\pi^2} + (\ell_1 L_1)^2 + \cdots + (\ell_j L_j)^2\right)^{\frac{j+1}{4}}}. \tag{B.9}$$

The Casimir energy is proportional to the finite part of (B.7) as $\lambda \to 0$. The first term in brackets in (B.7) is identical to the derivative of the first term in $\Lambda_j(q,\lambda)$ given by (15). Therefore the result (28) from section 2 is directly applicable, i.e.

$$\lim_{\lambda\to 0} \partial_\lambda \Lambda_j^{\text{finite}}(\lambda) = \frac{L_1 \cdots L_j}{(L_{j+1})^{j+1}} \left(\Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j\right), \tag{B.10}$$

where the remainder term is given by

$$R_j \equiv \lim_{\lambda\to 0} \partial_\lambda R_j(\lambda)$$

$$= \sum_{n=1}^{\infty} \underset{\substack{l_i=-\infty \\ i=1,\ldots,j}}{\sum_{}^{\infty}}{}' \frac{2(nL_{j+1})^{\frac{j+1}{2}}}{\pi[(\ell_1 L_1)^2 + \cdots + (\ell_j L_j)^2]^{\frac{j+1}{4}}} K_{\frac{j+1}{2}}\left(\frac{2\pi n}{L_{j+1}} \sqrt{(\ell_1 L_1)^2 + \cdots + (\ell_j L_j)^2}\right). \tag{B.11}$$

Our final Casimir energy expression for periodic boundary conditions is then given by

$$E_{p_{L_1\cdots L_d}}(d) = -\pi v \sum_{j=0}^{d-1} \lim_{\lambda\to 0} \partial_\lambda \Lambda_j^{\text{finite}}(\lambda)$$

$$= -\pi v \sum_{j=0}^{d-1} \frac{L_1 \cdots L_j}{(L_{j+1})^{j+1}} \left(\Gamma\left(\frac{j+2}{2}\right) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j\right)$$

$$= v \left(\frac{-\pi}{6L_1} - \frac{L_1}{L_2^2}\frac{\zeta(3)}{2\pi} - \frac{L_1 L_2}{L_3^3}\frac{\pi^2}{90} + \cdots - R_1 \frac{\pi L_1}{L_2^2} - R_2 \frac{\pi L_1 L_2}{L_3^3} + \cdots\right) \tag{B.12}$$

where the remainder R_j is given by (B.11) (note that R_j is zero when $j = 0$). Equation (B.12) is a highly compact way to express the Casimir energy for arbitrary lengths. As in section 2 it is split into two terms: an analytical part and a remainder. The same physical interpretation follows: the analytical part is a sum of parallel plate terms. Equation (B.12) is valid for any lengths and we know the result should be invariant under a permutation of the lengths. However, the two terms separately are not invariant, only their sum. We naturally want to label the lengths such that the remainder term lives up to its name. This can be accomplished if the largest length is labelled L_1, the next largest length L_2, i.e. $L_1 \geqslant L_2 \geqslant L_3 \cdots$. Then the Bessel function decreases exponentially fast and the remainder is small. If q

dimensions are large and $d - q$ dimensions have equal length L, equation (B.12) for $E_p(d)$ and equation (B.11) for the remainder R_j reduce to the results of section 2, i.e. $E_p(q, d)$ given by (30) and $R_j(q, d)$ given by (29) respectively.

The Neumann (N) and Dirichlet (D) cases can be obtained via simple permutations of the periodic case. The operator relations for Neumann and Dirichlet are $\sum_0^\infty \to \frac{1}{2}\left(\sum_{-\infty}^\infty +1\right)$ and $\sum_1^\infty \to \frac{1}{2}\left(\sum_{-\infty}^\infty -1\right)$ respectively. Applying the operator d times while keeping each sum distinct because of different lengths and multiplying the final result by $\frac{1}{2}$ yields the Neumann and Dirichlet energies

$$E_{N,D} = \frac{1}{2^{d+1}} \sum_{m=1}^{d} \sum_{(k_1,\ldots,k_m)} (\pm 1)^{d+m} E_{p_{k_1\ldots k_m}}(m), \tag{B.13}$$

where the $(+)$ is for Neumann and the $(-)$ for Dirichlet. The sum is over all sets $(k_1, \ldots, k_m)$ with $k_1 < k_2 < \cdots < k_m$ (the k's run from 1 to d). $E_{p_{k_1\ldots k_m}}(m)$ is the periodic energy (B.12) replacing d by m and L_1 by L_{k_1}, L_2 by L_{k_2}, etc.

Appendix C. Remainder term $R_d(s)$ for Epstein zeta function

We derive in this appendix a convenient form for the remainder $R_d(s)$ in terms of sums of Bessel and gamma functions. We begin with the expression for the remainder $R_d(s)$ given by (62)

$$R_d(s) \equiv \sum_{n_1,\ldots,n_{d-1}=1}^{\infty} \frac{-1}{2} \sum_{\nu=0}^{\infty} \int_0^1 B_2(x) \frac{\partial^2}{\partial x^2} \frac{1}{((x+\nu)^2 + n^2)^s} \, dx, \tag{C.1}$$

where

$$n^2 \equiv n_1^2 + \cdots + n_{d-1}^2. \tag{C.2}$$

We now follow similar procedures as those employed in appendix A for $R_j(q)$. To avoid being repetitive, we skim through details already discussed in appendix A.

The term $x + \nu$ is continuous and runs from 0 to ∞. We drop the sum over ν, replace $x + \nu$ by x and integrate from 0 to ∞ instead of 0 to 1. We replace $B_2(x) = x^2 - x + 1/6$ by its Fourier expansion (A.5), i.e.

$$x^2 - x + 1/6 = \sum_{\ell=1}^{\infty} \frac{\cos(2\pi \ell x)}{\ell^2 \pi^2}. \tag{C.3}$$

We can therefore make the following replacement in (C.1):

$$\sum_{\nu=0}^{\infty} \int_0^1 B_2(x) \frac{\partial^2 f(x+\nu)}{\partial x} \, dx \to \sum_{\ell=1}^{\infty} \frac{1}{\ell^2 \pi^2} \int_0^{\infty} \cos(2\pi \ell x) \frac{\partial^2 f(x)}{\partial x} \, dx \tag{C.4}$$

where $f(x)$ is the function in (C.1) with ν omitted, i.e.

$$f(x) = \frac{1}{(x^2 + n^2)^s} \tag{C.5}$$

The function $f(x)$ has the following properties:

$$\lim_{x \to 0} \frac{\partial f(x)}{\partial x} = 0, \qquad \lim_{x \to \infty} \frac{\partial f(x)}{\partial x} = 0, \qquad \lim_{x \to \infty} f(x) = 0. \tag{C.6}$$

With the above properties of f, (C.4) reduces to the same expression (A.9) obtained in appendix A:

$$\sum_{\nu=0}^{\infty} \int_0^1 B_2(x) \frac{\partial^2 f(x+\nu)}{\partial x} \, dx \to -4 \sum_{\ell=1}^{\infty} \int_0^{\infty} \cos(2\pi \ell x) f(x) \, dx. \tag{C.7}$$

After substituting (C.7) into (C.1) we obtain $R_d(s)$ in the following form:

$$R_d(s) = \sum_{n_1,\ldots,n_{d-1}=1}^{\infty} \sum_{\ell=1}^{\infty} \int_0^{\infty} 2\cos(2\pi\ell x) \frac{1}{(x^2+n^2)^s}\,\mathrm{d}x. \qquad \text{(C.8)}$$

The integral can be expressed in terms of Bessel functions, i.e.

$$\int_0^{\infty} 2\cos(2\pi\ell x) \frac{1}{(x^2+n^2)^s}\,\mathrm{d}x = \frac{2}{\sqrt{\pi}}\Gamma(1-s)\sin(\pi s)K_{s-1/2}(2\pi\ell n)\left(\frac{\pi\ell}{n}\right)^{s-1/2}. \qquad \text{(C.9)}$$

Our final expression for $R_d(s)$ is then

$$R_d(s) = \sum_{n_1,\ldots,n_{d-1}=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{2}{\sqrt{\pi}}\Gamma(1-s)\sin(\pi s)K_{s-1/2}(2\pi\ell n)\left(\frac{\pi\ell}{n}\right)^{s-1/2}, \qquad \text{(C.10)}$$

where $n \equiv \sqrt{n_1^2 + \cdots + n_{d-1}^2}$.

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